CHAPTER 6

BUCKLING OF ISOTROPIC COLUMNS AND PLATES

6.1 Derivation of the Plate Governing Equations for Buckling

The governing equations for a thin plate subjected to both in-plane and lateral loads have been derived previously. In those equations, there was one governing equation describing the relationship between the lateral deflection and the laterally distributed loading,

$$D\nabla^4 w = p(x, y)$$

and other equations dealing with in-plane displacements, related to in-plane loads

$$\nabla^4 u_0 = \nabla^4 v_0 = 0.$$

As discussed previously, the equations involving lateral displacements and lateral loads is completely independent (uncoupled) from those involving the in-plane loadings and in-plane displacements.

However, it is true that when in-plane loads are compressive, upon attaining certain discrete values, these compressive loads do result in producing lateral displacements. Thus, there does occur a coupling between in-plane loads and lateral displacements, w. As a result, a more inclusive theory must be developed to account for this phenomenon, which is called *buckling* or *elastic instability*.

Unlike in developing the governing plate equations in Chapter 1, wherein the development began with the three dimensional equations of elasticity, the following shall begin with looking at the in-plane forces acting on a plate element, in which the forces are assumed to be functions of the midsurface coordinates x and y, as shown in Figure 6.1.

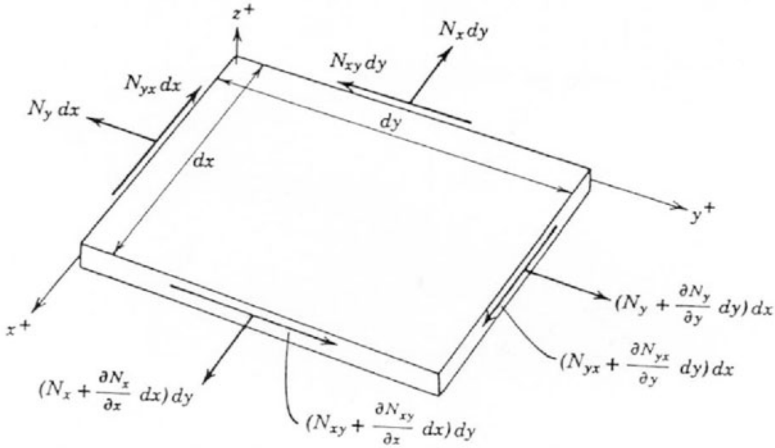

Figure 6.1. In-plane forces on a plate element.

Looking now at the plate element of Figure 6.2, viewed from the midsurface in the positive y direction, the relationship between forces and displacements is seen, when the plate is subjected to both lateral and in-plane forces, i.e., when there is a lateral deflection, w (note obviously that in the figure the deflection is exaggerated).

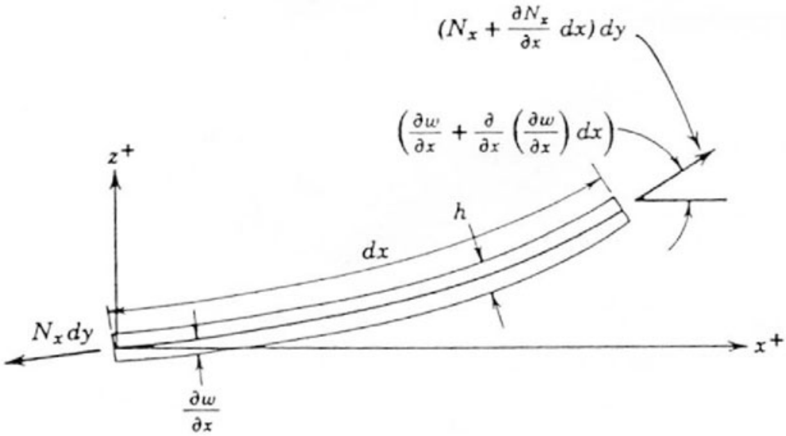

Figure 6.2. In-plane forces acting on a deflected plate element.

Hence, the z component of the N_x loading per unit area is, for small slopes (i.e., the sine of the angle equals the angle itself in radians):

$$\frac{1}{\mathrm{d}x\,\mathrm{d}y}\left[\left(N_x + \frac{\partial N_x}{\partial x}\,\mathrm{d}x\right)\mathrm{d}y\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\,\mathrm{d}x\right) - N_x\,\mathrm{d}y\frac{\partial w}{\partial x}\right]$$

Neglecting terms of higher order, the force per unit planform area in the z direction is seen to be

$$N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x}. \tag{6.1}$$

Similarly, the z component of the N_y force per unit planform area is seen to be

$$N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y}. \tag{6.2}$$

Finally to investigate the z component of the in-plane resultants N_{xy} and N_{yx},

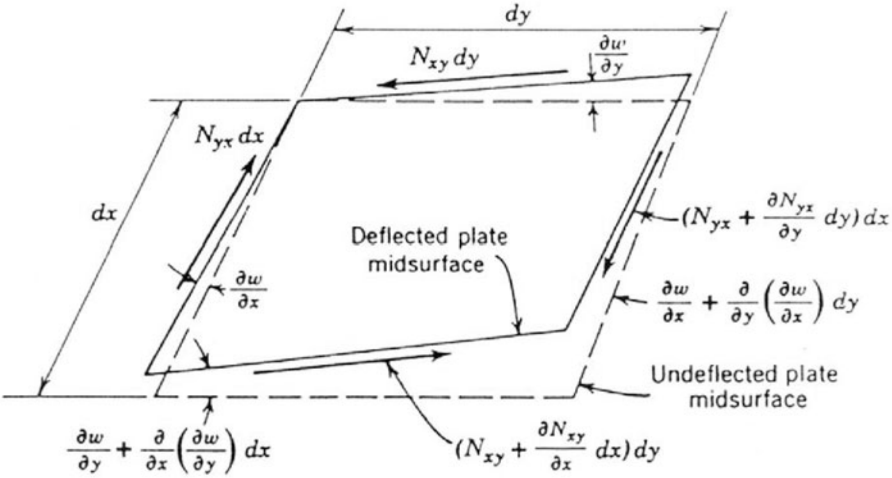

Figure 6.3. In-plane shear forces acting on a deflected plate element.

Hence, the z component per unit area of the in-plane shear resultant is:

$$\begin{aligned} \frac{1}{\mathrm{d}x\,\mathrm{d}y}\Bigg\{ &\left(N_{xy} + \frac{\partial N_{xy}}{\partial x}\,\mathrm{d}x\right)\left(\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\,\mathrm{d}x\right)\mathrm{d}y \\ &+ \left(N_{yx} + \frac{\partial N_{yx}}{\partial y}\,\mathrm{d}y\right)\left(\frac{\partial w}{\partial x} + \frac{\partial^2}{\partial x\,\partial y}\,\mathrm{d}y\right)\mathrm{d}x \\ &- N_{xy}\frac{\partial w}{\partial y}\,\mathrm{d}y - N_{yx}\frac{\partial w}{\partial x}\,\mathrm{d}x \Bigg\}. \end{aligned}$$

Neglecting higher order terms, this result in

$$N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y}+N_{yx}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial N_{yx}}{\partial y}\frac{\partial w}{\partial x}. \tag{6.3}$$

With all the above z components of forces per unit area, the governing plate equation can be modified to include the effect of these in-plane forces on the governing plate equations.

$$\begin{aligned} D\nabla^4 w &= p(x,y)+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} \\ &+\frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y}+\frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial N_{yx}}{\partial y}\frac{\partial w}{\partial x}. \end{aligned} \tag{6.4}$$

However, from in-plane force equilibrium, it is remembered from Equations (2.17) and (2.18), assuming no applied surface shear stresses, that

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=0, \qquad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0 \qquad (6.5), (6.6)$$

Substituting these into the expression above, the final form of the equation is found to be:

$$D\nabla^4 w = p(x,y)+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}. \tag{6.7}$$

Likewise, this governing plate equation can be reduced to the governing equation for a beam column by multiplying (6.7) by b (the width of the beam) and letting $\partial(\ )/\partial y = 0$, $\nu = 0$, $\overline{P} = -bN_x$ and $q(x) = bp(x)$, to provide

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+k^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=\frac{q(x)}{\mathrm{EI}} \quad \text{where} \quad k^2=\overline{P}/\mathrm{EI}. \tag{6.8}$$

It should be noted that the load $\overline{P}$ defined above is an in-plane load which when positive produces compressive stresses, which differs from the convention used elsewhere throughout this text. However, it is commonly used in the literature on buckling, is convenient, so herein is described as a barred quantity.

6.2 Buckling of Columns Simply Supported at Each End

Solving Equation (6.8) by methods described previously, the solution can be written as:

$$w(x) = A\cos kx + B\sin kx + C + Ex + w_p(x) \tag{6.9}$$

where $w_p(x)$ is the particular solution for the loading $q(x)$. Consider, for example, the case wherein $q(x) = 0$, and the column is simply supported at each end. The boundary conditions, at $x = 0$, L, are then

$$\begin{gathered} w(0) = w(L) = 0 \\ M_x\begin{pmatrix} L \\ 0 \end{pmatrix} = -\mathrm{EI}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \quad \text{or} \quad \frac{\mathrm{d}^2 w(0)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w(L)}{\mathrm{d}x^2} = 0. \end{gathered} \tag{6.10}$$

From the first boundary condition $A + C = 0$, from the third $A = 0$; hence, $C = 0$ also. From the second boundary condition $B\sin kL + EL = 0$, and from the fourth boundary condition

$$Bk^2\sin kL = 0 = \frac{B\overline{P}}{\mathrm{EI}}\sin kL = 0. \tag{6.11}$$

Note that in Equation (6.11) when $kL \neq n\pi$, then $B = E = 0$; when $kL = n\pi$, then $E = 0$, $B \neq 0$ but is indeterminate and

$$\overline{P} = n^2\pi^2\frac{\mathrm{EI}}{L^2}. \tag{6.12}$$

It is thus seen that for most values of $\overline{P}$, the axial compressive loading, the lateral deflection w is zero ($A = B = C = E = 0$), and the in-plane and lateral forces and responses are uncoupled. However, for a countable infinity of discrete values of P, there is a lateral deflection, but it is of an indeterminate magnitude. Mathematically, this is referred to as an *eigenvalue problem* and the discrete values given in (6.12) are called *eigenvalues*. The resulting deflections, in this case, are

$$w(x) = B\sin kx$$

and are called *eigenfunctions*.

The natural vibration of elastic bodies are also eigenvalue problems, where in that case the natural frequencies are the eigenvalues and the vibration modes are the eigenfunctions. This is treated in the next chapter.

As to buckling, looking at Equation (6.12), as $\overline{P}$ increases, it is clear that the lowest buckling load occurs when $n = 1$, and at that particular load, the column will either inelastically deform and strain harden, or the column will fracture. Hence, $n > 1$ has no physical significance. The load

$$\overline{P} = \pi^2 \frac{\mathrm{EI}}{L^2} \tag{6.13}$$

is therefore the critical buckling load for this column for these boundary conditions. In this particular case the buckling load is called the Euler buckling load, since the Swiss mathematician was the first to solve the problem successfully.

Another way to phrase the buckling problem is exemplified by solving Equation (6.8), letting $q(x) = q_0 =$ constant. The resulting particular solution, in this case, is $q_0 = x^2/2P$. If the column is simply supported, solving the boundary value problem for the lateral deflection, results in

$$w(x) = \frac{q_0}{Pk^2 \sin kL}\left[\cos kx \sin kL - \cos kL - Lx \sin kL + k^2 x^2 \sin kL\right]. \tag{6.14}$$

In Equation (6.14), the solution of a boundary value problem, when the axial load $\overline{P}$ has values given in (6.12) wherein $\sin kL = 0$, then $w(x)$ goes to infinity, or, more properly, since we have a small deflection linear mathematical model, $w(x)$ becomes indefinitely large.

Hence, whether we solve for the homogeneous solution of Equation (6.8), resulting in an eigenvalue problem, or we solve the nonhomogeneous Equation (6.8), resulting in a boundary value problem, the results are identical, when $\overline{P}$ has values given by (6.12), or physically where $\overline{P}$ attains the value given by (6.13), the column 'buckles'.

Note also that the buckling load, Equation (6.13), is not affected by any lateral load $q(x)$. The physical significance of a lateral load $q(x)$, however, is that the beam-column may deflect sufficiently, due to both the lateral and in-plane compressive loads, that the resulting curvature would cause bending stresses which in addition to the compressive stresses may fracture or yield the column at a load less than or prior to attaining the buckling load.

These elastic stability considerations are very important in analyzing or designing any structure in which compressive stresses result from the loading, because in addition to insuring that the structure is not merely overstressed or overdeflected, in this case a new failure mode has been added, i.e., buckling.

6.3 Column Buckling with Other Boundary Conditions

From the previous section, the critical compressive buckling load $\overline{P}_{cr}$ is given as

$$\overline{P}_{cr} = \pi^2 \frac{EI}{L^2} \tag{6.15}$$

Numerous other texts derive critical buckling loads for columns with other boundary conditions, [6.1] through [6.4], and [12.2].

For ease of use in analysis and design, but without derivations, the following column buckling equations are listed for the other classical boundary conditions.

Column with both ends clamped

$$\overline{P}_{cr} = 4\pi^2 \frac{EI}{L^2}. \tag{6.16}$$

Column with one end clamped and the other simply supported

$$\overline{P}_{cr} = \frac{\pi^2 EI}{(0.669L)^2}. \tag{6.17}$$

Column with one end clamped and the other end free

$$\overline{P}_{cr} = \frac{\pi^2 EI}{4L^2}. \tag{6.18}$$

6.4 Buckling of Isotropic Rectangular Plates Simply Supported on All Four Edges

Plate buckling qualitatively is analogous to column buckling, except that the mathematics is more complicated, and the conditions that result in the lowest eigenvalue (the actual buckling load) are not so lucid in many cases.

Whenever the in-plane forces are compressive, and are more than a few percent of the plate buckling loads (to be defined later), Equation (6.7) must be used rather than Equation (3.1) in the analysis of plates.

For the plate, just as the case of the beam-column, since the in-plane load that causes an elastic stability is not dependent upon a lateral load, to investigate the elastic stability we shall assume $p(x, y) = 0$ in Equation (6.7).

Consider, as an example, a simply supported plate subjected to constant in-plane loads N_x and N_y (let $N_{xy} = 0$), as shown in Figure 6.4.

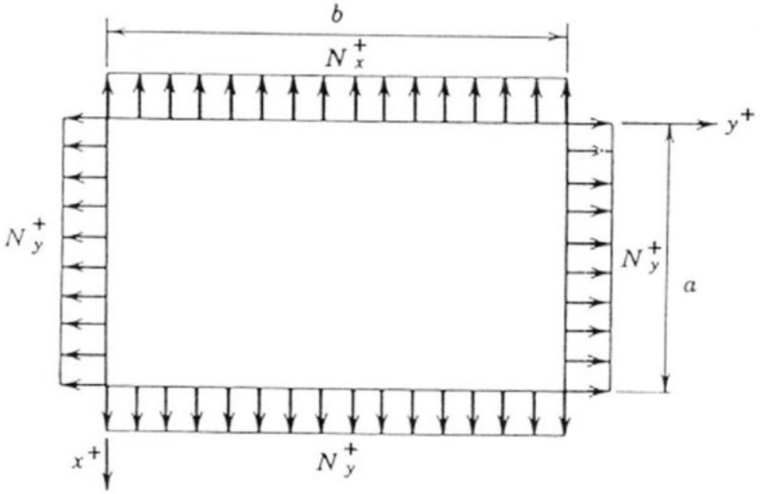

Figure 6.4. Rectangular plate subjected to in-plane loads.

Assume the solution of Equation (6.7) to be of the Navier form

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{6.19}$$

Substituting (6.19) into (6.7), it is convenient to define α here to be

$$\alpha = N_y / N_x . \tag{6.20}$$

The solution to the eigenvalue problem is found to be

$$N_{x\,\mathrm{cr}} = -D\pi^2 \frac{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\alpha\right]}. \tag{6.21}$$

Here the subscript cr denotes that this is a critical load situation – the plate buckles. Also note that in (6.21) N_x is negative, i.e., a load that causes compressive stresses.

Equation (6.21) is the complete set of eigenvalues for the simply supported plate, analogous to Equation (6.12) for the column. In other words for these discrete values of N_x and N_y, Equation (6.7) has nontrivial solutions wherein the lateral deflection is given by (6.19); for other values $w(x, y) = 0$.

Since we know that as the load increases, the plate will buckle at the lowest buckling load (or eigenvalue) and all the rest of the eigenvalues have no physical meaning. So it is necessary to determine what values of the integers *m* and *n* (the number of half sine waves) make N_x a minimum.

Defining the length to width ratio of the plate to be $r = a/b$ Equation (6.21) can be rewritten as

$$N_x = -\frac{D\pi^2}{a^2}\frac{[m^2 + n^2 r^2]^2}{[m^2 + n^2 r^2 \alpha]}. \tag{6.22}$$

Note if in Equation (6.22) $\alpha = 0$, $r = 1$ and $m = n = 1$, then

$$N_x = -\frac{4\pi^2 D}{a^2}. \tag{6.23}$$

Note the similarities between Equations (6.23) and (6.13).

The question remains; given a combination of N_x and N_y loadings, and a given

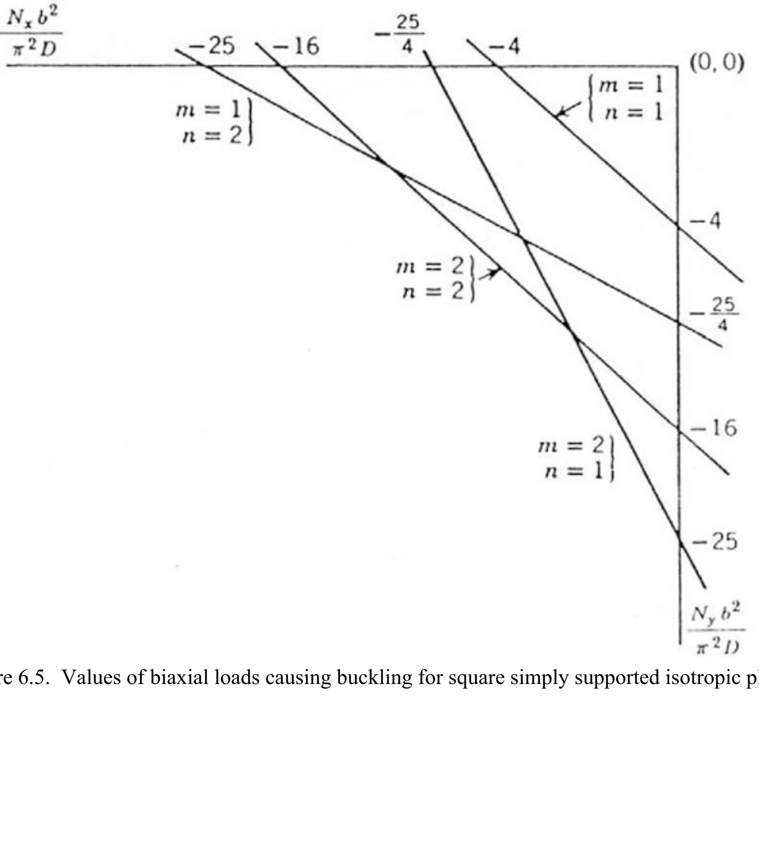

Figure 6.5. Values of biaxial loads causing buckling for square simply supported isotropic plate.

geometry r, what values of m and n provide the lowest buckling loads. One can make a plot such as Figure 6.5 above from manipulating Equation (6.22) (which is not shown to scale) for a square plate ($a = b$, $r = 1$).

It is seen from Figure 6.5 that for such a square plate, simply supported on all four edges, the plate will always buckle into a half sine wave ($m = n = 1$) under any combination of N_x and/or N_y, since that line is always closest to the origin, hence, the lowest buckling load situation.

Next consider a plate under an in-plane load in the x direction only, so $N_y = 0$, and $\alpha = 0$. In this case, Equation (6.21) can be written as

$$N_{x\,cr} = -\frac{D\pi^2 a^2}{m^2}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^2. \tag{6.24}$$

The loaded plate is shown in Figure 6.6.

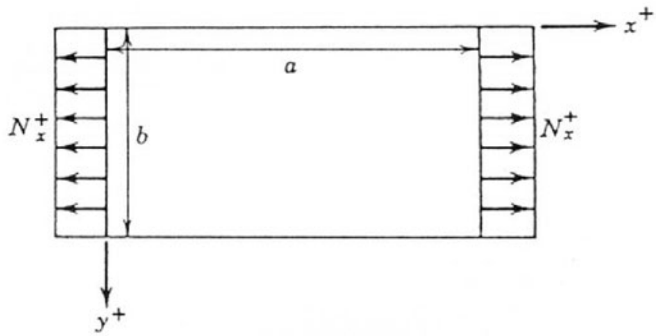

Figure 6.6. Plate subjected to in-plane load in the x direction.

Examination of Equation (6.24) shows that the first term is merely the Euler column load (6.13) for a column of unit width, including Poisson ratio effects. The second term clearly shows the buckle resisting effect providing by the simply supported side edges, and this effect diminishes as the plate gets wider, i.e., as b increases. In fact as $b \to \infty$, (6.24) shows that the plate acts merely as an infinity of unit width beams, simply supported at the ends, and because they are 'joined together', the Poisson ratio effect occurs, i.e., D instead of EI appears.

It is obvious from Equation (6.24) that the minimum values of N_x occurs when $n = 1$, since n appears only in the numerator. Thus for an isotropic plate, simply supported on all four edges, subjected only to an uniaxial in-plane load the buckling mode given by (6.19) will always be one half sine wave [sin(y/b)] across the span, regardless of the length or width of the plate.

Thus, since $n = 1$, Equation (6.24) can be written as

$$N_{x\,cr} = -\frac{D\pi^2}{b^2}\left(\frac{m}{r}+\frac{r}{m}\right)^2 \tag{6.25}$$

where it is remembered that $r = a/b$, termed the aspect ratio.

Now if $a < b$ (the plate wider than it is long), the second term is always less than the first, hence, the minimum value of N_x is always obtained by letting $m = 1$. Hence for $a \le b$, the buckling mode for the simply supported plate is always

$$w(x,y) = A_{11}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{\mathrm{b}}\right). \tag{6.26}$$

In that case,

$$N_{x\,cr} = -\frac{D\pi^2}{b^2}\left(\frac{1}{r}+r\right)^2. \tag{6.27}$$

To find out at what aspect ratio r, that N_x is truly a minimum, let

$$\frac{\mathrm{d}N_{x\,\mathrm{cr}}}{\mathrm{d}r} = 0 = -\frac{2D\pi^2}{b^2}\left(\frac{1}{r}+r\right)\left(-\frac{1}{r^2}+1\right).$$

Therefore $r = 1$ provides that minimum value. Hence for $m = 1$, N_x is a minimum when $a = b$. Under that condition, from (6.27)

$$N_{x\,\mathrm{cr}\,a=b} = -\frac{4D\pi^2}{b^2} = -\frac{4D\pi^2}{a^2}. \tag{6.28}$$

Comparing this with the Euler buckling load of (6.13) for a simply supported column, it is seen that the continuity of a plate and the support along the sides of the plate provide a factor of at least 4 over the buckling of a series of strips (columns) that are neither continuous nor supported along the unloaded edges.

Now as the length to width ratio increases, as a/b increases, the buckling load (6.27) will increase, and one can ask, will $m = 1$ always result in a minimum buckling load, or is there another value of m which will provide a lower buckling load as r increases (i.e., $N_{x\,\mathrm{cr}}\ (\mathrm{m}=2) \le N_{x\,\mathrm{cr}}\ (\mathrm{m}=1)$ for some value of r?)

Mathematically, this can be phrased as the following, using (6.25):

$$\left(\frac{m}{r}+\frac{r}{m}\right)^2 \overset{?}{\le} \left(\frac{m-1}{r}+\frac{r}{m-1}\right)^2.$$

This states the condition under which the plate of aspect ratio r will buckle in m half sine waves in the loaded direction rather than m – 1 sine waves. Manipulating this inequality results in

$$m(m-1) \leq r^2. \tag{6.29}$$

Equation (6.29) states that the plate will buckle in two half sine waves in the axial direction rather than one when $r \geq \sqrt{2}$. The plate will buckle in three half sine waves in the axial direction rather than 2, when $r \geq \sqrt{6}$, etc.

Again one can ask that when the plate buckles into m = 2 configuration, does a minimum buckling load occur, if so at what r and what is $N_{x\,\mathrm{cr\,(min)}}$?

From Equation (6.25)

$$\frac{\mathrm{d}N_{x\,\mathrm{cr}}}{\mathrm{d}r}(\mathrm{m}=2) = 0 = -\frac{2D\pi^2}{b^2}\left(\frac{2}{r}+\frac{r}{2}\right)\left(-\frac{2}{r^2}+\frac{1}{2}\right) = 0$$

or $r^2 = 4$, $r = 2$.

$$N_{x\,\mathrm{cr\,(min)}} = -\frac{4D\pi^2}{b^2} \quad \text{for} \quad m = 2. \tag{6.30}$$

This is the same value as is given in Equation (6.28) for m = 1. Proceeding with all values of r and m, the following graph can be drawn, which clearly shows the results (Figure 6.7).

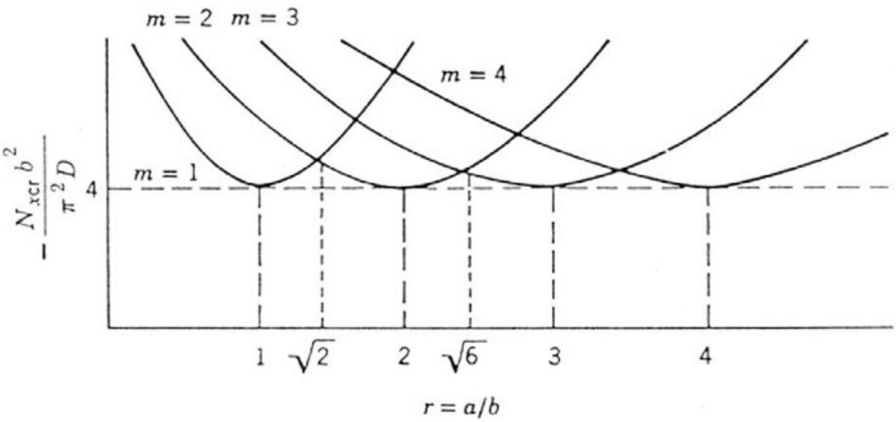

Figure 6.7. Buckling load as a function of aspect ratio for a simply supported isotropic plate.

Hence knowing the value of r, the figure provides the actual value of N_x and the corresponding value of the wave number m in the load direction. However, in practice for $r > 1$, universally one simply uses Equation (6.28) or (6.30) for the buckling load.

However, looking more closely at Equation (6.29), as m increases we see

$$m(m-1) \rightarrow m^2 = r^2 \quad \text{or} \quad m = r = a/b.$$

This means that for long plates, the number of half sine waves of the buckles have lengths approximately equal to the plate width. Another way to stating it is that a long plate simply supported on all four edges and subjected to a uniaxial compressive load attempts to buckle into a number of square cells.

Remembering that $\sigma_x = N_x/h$, Equation (6.28) or (6.30) can be written as the following for $a/b \geq 1$,

$$\sigma_{\text{cr}} = -\frac{\pi^2 E}{3(1-\nu^2)}\left(\frac{h}{b}\right)^2. \tag{6.31}$$

6.5 Buckling of Isotropic Plates with Other Loads and Boundary Conditions

The solution to the buckling of flat isotropic plates simply supported on all four sides subjected to uniaxial uniform compressive in-plane loads has been treated in detail. However, for many other boundary conditions, simple displacement functions like Equation (6.19) do not exist, and in some cases analytical, exact solutions analogous to Equations (6.21) and (6.31) have not been found. In those cases approximate solutions have been found using energy methods, which will be discussed in Chapters 8 and 9. These have been catalogued by Gerard and Becker [6.3] among others, and are presented in Figure 6.8 and k_c, given in the following equations:

$$\sigma_{x\,\text{cr}} = -\frac{k_c \pi^2 E}{12(1-\nu^2)}\left(\frac{h}{b}\right)^2; \quad N_{x\,\text{cr}} = -\frac{k_c \pi^2 D}{b^2} \tag{6.32}$$

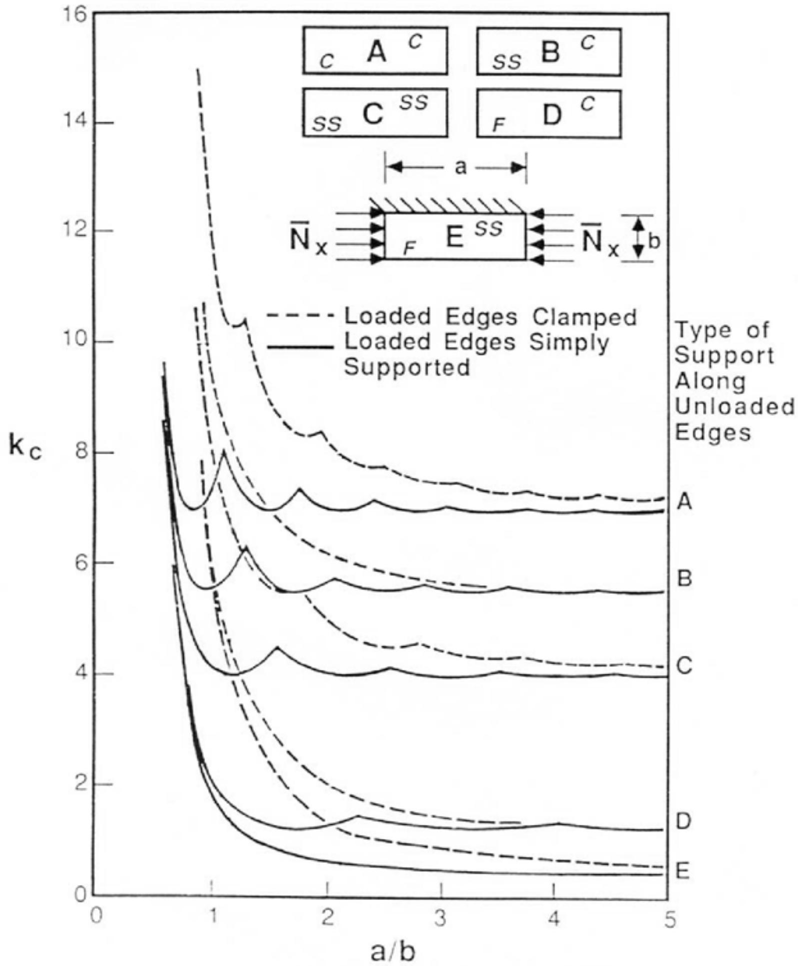

Figure 6.8. Compressive-buckling coefficients for flat rectangular isotropic plates.

In many practical applications, the edge rotational restraints lie somewhere between fully clamped and simply supported along the unloaded edges. For the case of the loaded edges simply supported, the buckling coefficient, k_c, of Equation (6.32) are also given by Gerard and Becker [6.3] as shown in Figure 6.9. The unloaded edge restraint, ε, is zero for simply supported edges and infinity for full clamping. Values in between these extremes require engineering judgment.

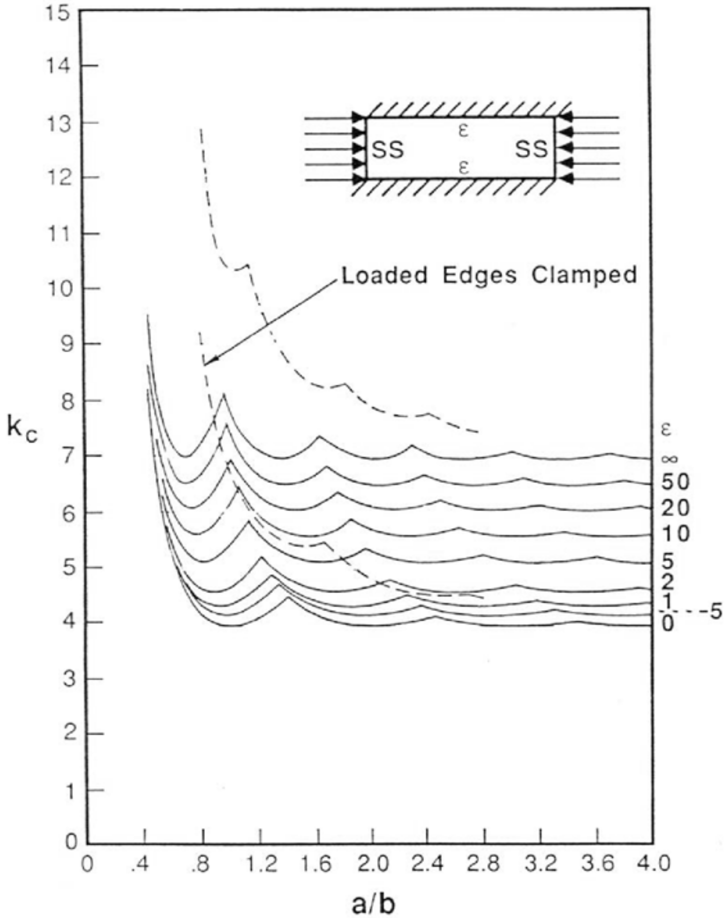

Figure 6.9. Compressive-buckling-stress coefficient of isotropic plates as a function of *a*/*b* for various amounts of edge rotational restraint.

For in-plane shear loading, the critical shear stress is given by the following equations:

$$\tau_{\mathrm{cr}} = \frac{K_s \pi^2 E}{12(1-\nu^2)}\left(\frac{h}{b}\right)^2; \quad N_{xy\,\mathrm{cr}} = \frac{K_s \pi^2 D}{b^2} \tag{6.33}$$

where K_s is given in Figure 6.10 for various boundary conditions [6.3].

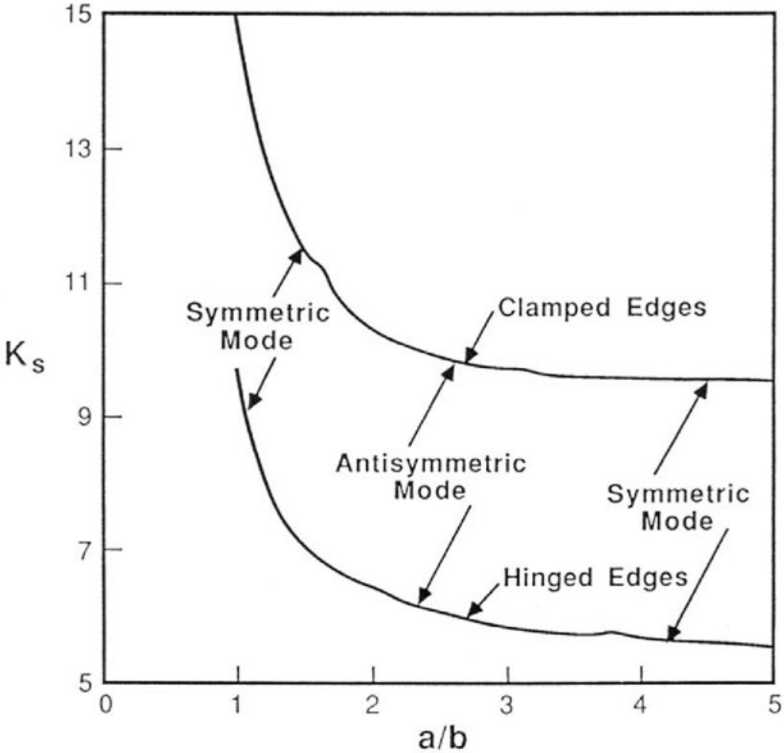

Figure 6.10. Shear-buckling-stress coefficient of isotropic plates as a function of *a*/*b* for clamped and hinged edges.

For rectangular plates subjected to in-plane bending loads, the following equation is used to determine the stress value for the buckling of the plate shown in Figure 6.11.

$$\sigma_B = \frac{k_b \pi^2 E}{12(1-\nu^2)}\left(\frac{h}{b}\right)^2 \tag{6.34}$$

where again ε is the value of the edge constraint as discussed previously.

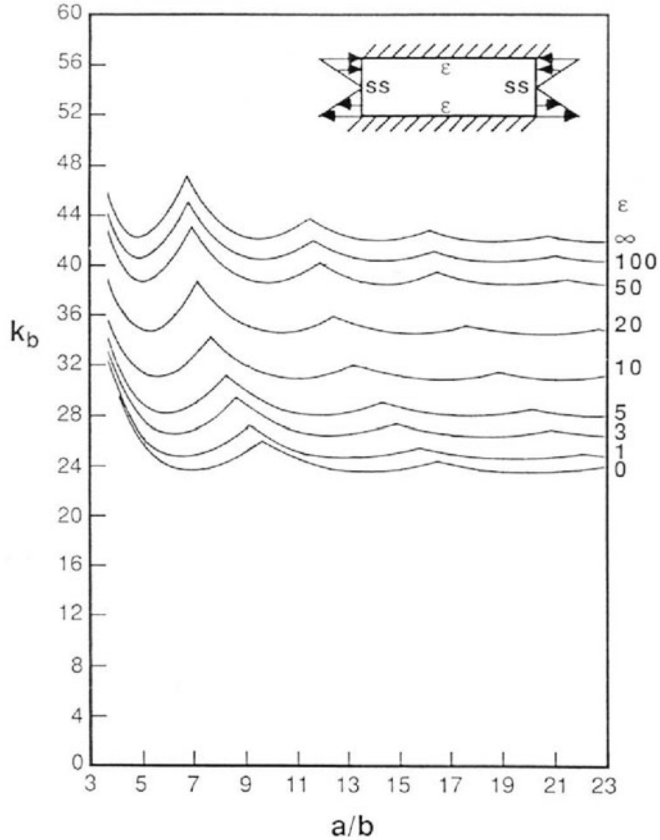

Figure 6.11. Bending-buckling coefficient of isotropic plates as a function of *a*/*b* for various amounts of edge rotational restraint.

6.6 The Buckling of an Isotropic Plate on an Elastic Foundation Subjected to Biaxial In-Plane Compressive Loads

It is important to consider that besides overall buckling of the entire plate, it is possible that a sandwich face plate may buckle, due to loads applied to the face. In this case the plate can be considered to be supported on a uniform elastic foundation, namely the core. In such a case the buckling equation for this phenomenon is

$$D\nabla^4 w + kw + \overline{N}_x \frac{\partial^2 w}{\partial x^2} + \overline{N}_y \frac{\partial^2 w}{\partial y^2} = 0 \tag{6.35}$$

where D is the flexural stiffness of the face plate, w is the lateral displacement of the face plate, k is the foundation modulus in force/unit area/unit deflection, and $\overline{N}_x, \overline{N}_y$ are the compressive loads per unit width in the subscripted direction (i.e., $\overline{N}_x = -N_x$, etc.) acting on that particular face plate.

Considering this localized buckling phenomenon, is has been found that the plate boundary conditions at the outer plate edges do not affect the buckling load. Therefore, for analytical simplicity, assume simply-supported edges on all four sides. Therefore, the Navier approach may be used for the solution, with the lateral deflection assumed to be

$$w(x,y) = A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{6.36}$$

where A_{mn} is the deflection amplitude, a is the plate dimension in the x-direction, and b is the plate dimension in the y-direction.

For simplification, let $\phi = \overline{N}_y / \overline{N}_x$ and $r = a/b$. Substituting Equation (6.36) into (6.35) and using the above

$$\overline{N}_{x\mathrm{cr}} = \frac{(\pi^4 D/a^4)(m^2 + n^2 r^2)^2 + k}{(\pi^2/a^2)(m^2 + n^2 r^2 \phi)} \tag{6.37}$$

If $\phi = 1$, and $r = 1$, then the response is independent of direction. When the in-plane loads are caused by the cooling of a sandwich plate wherein the coefficients of thermal expansion between face and core cause the face to be compressed, then $\phi = 1$. Further because the buckling is a localized phenomenon, one can let $r = 1$. Then Equation (6.37) may be written as

$$\overline{N}_{x\mathrm{cr}} = \frac{(\pi^4 D/a^2)(m^2 + n^2 r^2)^2 + ka^2}{\pi^2(m^2 + n^2)} \tag{6.38}$$

First it is seen that the minimum value of $\overline{N}_x$ will occur when $m = n = 1$, therefore

$$\overline{N}_x = \frac{(4\pi^4 D/a^2) + ka^2}{2\pi^2} \tag{6.39}$$

To find the dimension a resulting in a minimum value of $\overline{N}_{x\mathrm{cr}}$, set $\partial \overline{N}_{x\mathrm{cr}} / \partial a = 0$, with the result that

$$a = 2^{1/2}\pi\left(\frac{D}{k}\right)^{\frac{1}{4}} \tag{6.40}$$

This is the half wavelength of the buckle that will occur, and it can be determined that this is a localized buckle in a reasonably sized face plate. Substituting Equation (6.40) into (6.39) results in

$$\overline{N}_{x\mathrm{cr}} = 2(kD)^{1/2}$$

As defined, $\overline{N}_{x\mathrm{cr}}$ is a compressive force per unit width and equal to $\overline{N}_{y\mathrm{cr}}$, since $\phi = 1$, or in the usual notation, where $\overline{N}_i = -N_i$,

$$N_{x_{\mathrm{cr}}} = N_{y_{\mathrm{cr}}} = -2(kD)^{1/2} \tag{6.41}$$

The buckling stress in the face plate is therefore

$$\sigma_{\mathrm{cr}} = \frac{-2}{h}(kD)^{1/2} \tag{6.42}$$

It has been found that in the fabrication of some sandwich plates, because of the cooling down subsequent to joining, the faces to the core in a rolling operation, differential thermal contractions caused sufficiently high compressive stresses in the faces to cause thermal buckling of the sandwich faces.

6.7 References

6.1. Timoshenko, S. and Gere, J. (1961) *Elastic Stability*, McGraw-Hill Book Co., Inc., 2nd Edition.
6.2. Bleich, H.H. (1952) *Buckling of Metal Structures*, McGraw-Hill Book Co., Inc.
6.3. Gerard, G. and Becker, H. (1957) *Handbook of Structural Stability*, *Part 1 – Buckling of Flat Plates*, NACA TN 3781.
6.4. Jones, R.M. (2004) *Buckling of Bars, Plates and Shells*, R.T. Edwards, Flourtown, PA.

6.8 Problems

6.1. In a plate clamped on all four edges, $\nu = 0.25$ and loaded in the x direction the critical buckling stress is given by (from Reference 7.1)

$$\sigma_{\mathrm{cr}} = -\frac{k\pi^2 D}{b^2 h} = -\frac{k\pi^2 E}{12(1-\nu^2)}\left(\frac{h}{b}\right)^2$$

where D is the flexural stiffness, b is the plate width, a is the plate length, and h is the plate thickness. k_c is given by

a/b	0.75	1.0	1.5	2.0	2.5	3.0
k	11.69	10.07	8.33	7.88	7.57	7.37

(a) Part of a support fixture for a missile launcher measure $45'' \times 15''$, and must support 145,000 lbs in axially compressive load. Its edges are all clamped. If the plate is composed of aluminum with $E = 10 \times 10^6$ psi, allowable 30,000 psi (both the tensile and compressive allowable stress is of magnitude 30,000 psi) and $\nu = 0.25$. What thickness is required to prevent buckling? What thickness is required to prevent overstressing?

(b) Suppose a steel plate of the same dimensions were used instead of the aluminum with the following properties: $E_{\text{steel}} = 30 \times 10^6$ psi, $\nu = 0.25$ and $\sigma_{\text{allowable}} = \pm 100{,}000$ psi. What thickness is needed to prevent buckling? Will the steel plate be overstressed?

(c) The density of steel is $0.283\ \text{lbs/in}^3$, the density of aluminum is $0.100\ \text{lbs/in}^3$. Which plate will be lighter?

6.2. A structural component in the interior of an underwater structure consists of a square plate of dimension a, simply supported on all four sides. If the component is subjected to in-plane compressive loads in both the x and y directions of equal magnitude, find N_x.

6.3. An aluminum support structure consists of a rectangular plate simply supported on all four edges is subjected to an in-plane uniaxial compressive load. If the length of the plate in the load direction is 4 feet, the width 3 feet, determine the minimum plate thickness to insure that the plate would buckling in the elastic range, if the material properties are $E = 10 \times 10^6$ psi, $\nu = 0.3$ and the compressive yield stress, $\sigma_y = 30{,}000$ psi.

6.4. A rectangular plate 4 feet × 2 feet is subjected to an in-plane compressive load N_x in the longer direction as shown in Figure 7.6. How much weight of plate can be saved by using a plate clamped on all four edges rather than having the plate simply supported on all four edges to resist the same compressive load $N_{x\,\text{cr}}$? Express the answer as a percentage.

6.5. An aluminum plate measure 6 feet × 3 feet, of thickness 0.1 inch is clamped on all four edges. Use the material properties in Problem 6.3 above.

(a) If it is subjected to a compressive in-plane load in the longer direction, what is the buckling stress?

(b) How much higher is the buckling stress compared to the same plate simply supported on all four edges?